ELSEVIER Journal of Geometry and Physics 25 (1998) 69–90

JOURNAL OF GEOMETRY AND PHYSICS

On the effective Yang–Mills Lagrangian and its equation of motion ★

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Received 8 January 1997

Abstract

The path integral quantization of gauge theories leads to an effective Lagrangian denoted by L_{eff}. Its associated equation of motion is of the form

$$d_A^* F_A + \tfrac{1}{2} J_A = 0,$$

where J_A is the vacuum polarization current arising from propagation of quantum fields around the background field A. In this paper we will give a more detailed description of the current J_A. © 1998 Elsevier Science B.V.

Subj. Class.: Quantum field theory

1991 MSC: 81T13, 83C45, 58D27, 58D30

Keywords: Riemannian four-manifolds; Gauge theory; Path-integral quantization; Yang–Mills Lagrangian; Quantum field theory; Vacuum polarization current

1. Introduction

The Yang–Mills equation is the equation of motion of a physical system in which matter fields are absent. Its Lagrangian is therefore of the most simplest form:

$$L_{\mathrm{YM}}(A) = \frac{1}{2} \int_M \| F_A \|^2 \; \mathrm{d}\, vol(M). \tag{1.1}$$

★ Work performed under the auspices of the GNSAGA of the CNR (National Research Council) of Italy and partially supported by the Ministero per la Ricerca Scientifica e Tecnologica within the projects *Geometria Reale e Complessa* and *Topologia*.

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PII S0393-0440(97)00022-3

Its Euler–Lagrange equation is

$$d_A^* F_A = 0. \tag{1.2}$$

A particular subspace of solutions of (1.2), namely the space of instantons, is extensively studied. It is worthwhile to note that there are also non-instanton solutions (see [4]). The moduli space $\mathcal{M}$ of instantons is useful to predict appearance (or disappearance) of instantons using for example the semiclassical approach (see [8]). For any kind of physical relevant information one has to quantize the gauge theory. One possibility is the Faddeev–Popov method which is based on the path-integral formalism (see for instance [17] or [18]). As an intermediate step one obtains a new Lagrangian, called *effective Lagrangian*.

It can be written in the following way:

$$L_{\text{eff}}(A, \eta, \bar{\eta}) = L_{\text{YM}}(A) + \int \langle \bar{\eta}, S\eta \rangle + GF(A, \eta, \bar{\eta}). \tag{1.3}$$

Here η and $\bar{\eta}$ are auxiliary fields, called "ghost" and "anti-ghost" fields. The term $GF(A, \eta, \bar{\eta})$ comes from a "gauge-fixing", i.e. from a local section in a fibre bundle. There will be no global section in general (see [19]). Since the path-integral "measure" is complex, the ghost and anti-ghost fields take their values in the odd part of a superspace and the integral $\int \langle \bar{\eta}, S\eta \rangle$ is understood in the sense of Berezin (see [1,11]). With this it is possible to interpret $\exp(\mathrm{i} \int \langle \bar{\eta}, S\eta \rangle)$ as the Berezin determinant of the operator S. Note that L_{eff} is no longer gauge invariant. There is a new type of symmetry called *BRS-symmetry* (see [17]). These can be interpreted as super-gauge transformations (see [9]). They became an interesting field to study, mainly from the algebraic point of view (see for instance [3] or [10]).

In this paper we quantize gauge theories using the real path-integral measure. Moreover, on the base manifold we take a Riemannian metric instead of a physically relevant Lorenzian metric (commonly known as euclidean quantum field theory).

One obtains the following effective Lagrangian:

$$L_{\text{eff}}(A) = L_{\text{YM}}(A) + \tfrac{1}{2}\zeta_A'(0), \tag{1.4}$$

where $\zeta_A(s)$ is the *zeta-function* of the self-adjoint positive elliptic operator $d_A^* d_A$ (we restrict our attention to irreducible connections A to obtain positivity of the Laplacian $d_A^* d_A$).

The associated equation of motion becomes

$$d_A^* F_A + \tfrac{1}{2} J_A = 0. \tag{1.5}$$

Note that (since $d_A^* d_A^* F_A = 0$)

$$d_A^* J_A = 0 \tag{1.6}$$

if the gauge-potential A satisfies (1.5). Physically, J_A can be considered as a current. It is the vacuum polarization current due to the propagation of quantum fields arising from the field A (see [5]). In Sections 3 and 4 we will study more precisely the current J_A by making use of the heat-kernel. In Section 3 we derive general formulas for the calculation of $\delta\zeta_A'(0)$ using standard techniques (see [2,6,8]). In particular, we will also point out the close relation to the second fundamental form of gauge orbits (see [12]). We state this in Corollary 3. In Section 4 we generalize a proof of Corrigan–Goddard–Osborn–Templeton (see [6, Section 2]) to obtain the following theorem which will be our main result.

Theorem 1. *Let $P \to M^4$ be a principal G-bundle over a closed connected orientable Riemannian 4-manifold M^4. Let A be an irreducible principal connection and let*

$$d_A^* d_A : \Omega^0(M, adP) \to \Omega^0(M, adP)$$

be its Laplacian with associated Green function

$$G_A(x, y) = S_A(x, y) + R_A(x, y),$$

where $R_A(x, y)$ is smooth, i.e. $R_A(x, y)$ is the regular part of $G_A(x, y)$.
Moreover, let

$$a_j(x, y) \in \Gamma(M \times M, \mathrm{End}(adP))$$

be the jth coefficient in the asymptotic heat-kernel expansion of $d_A^* d_A$.
If the scalar curvature of M is zero, then we have

$$J_A = -\frac{1 + C}{12\pi^2} d_A^* F_A + \frac{b - 4}{(4\pi)^2} d_A a_1 - d_A R_A,$$

where C is the Euler constant and

$$b = \tfrac{3}{2} - C + \log 4 \qquad (\textit{see Appendix A}).$$

We think of the group G to be the group SU(2). But the theorem holds for other groups too. The formula needs some explanation. It holds in $\Omega^1(M, \mathrm{End}(adP))$. By $F_A \in \Omega^2(M, adP)$ we denote the curvature of A, i.e. $d_A^* F_A \in \Omega^1(M, \mathrm{End}(adP))$. We think of $d_A^* F_A \in \Omega^1(M, \mathrm{End}(adP))$ as the endomorphism 1-form induced by $[d_A^* F_A(x), \cdot]$ in each fibre. Moreover, $d_A a_1$ is the "partial" covariant derivative applied to the first variable of $a_1(x, y)$ restricted to the diagonal, i.e.

$$d_A^x a_1(x, y)|_{y=x}.$$

Similarly,

$$d_A R_A(x) = d_A^x R_A(x, y)|_{y=x}.$$

The coefficient a_1 appears explicitely in the Green function (see Section 4):

$$G_A(x, y) = \frac{1}{(4\pi)^2}\left(\frac{4a_0(x, y)}{\rho^2(x, y)} - a_1(x, y)\log\rho^2(x, y)\right) + R_A(x, y).$$

The case treated in [6] is when $a_1 \equiv 0$. Their theorem then says that the polarization current J_A around an instanton depends only on the regular part of its Green function. Our hypothesis that the scalar curvature is zero is equivalent to

$$a_1(x, x) = 0$$

for every $x \in M$ (see for instance [2, p. 84] or [7]). Most of our arguments used to prove the above formula can be found in the literature. We are particularly indebted to [2,6,8,12,14].

2. Quantization of Yang–Mills functionals and the effective Lagrangian

The goal of this section is the derivation of a Lagrange functional by means of field quantization. For this we follow closely the paper of Groisser and Parker [8]. We recall first some basic notation. Let $P \to M^4$ be a principal G-bundle over a closed connected orientable Riemannian 4-manifold M. Our standard example will be an SU(2)-bundle. Let $\Gamma(AdP) = \mathcal{G}$ be the gauge group, i.e. $AdP = P \times_{Ad} G$. By $\mathcal{C}$ we denote the affine space of connections on P. An element $A \in \mathcal{C}$ determines a covariant derivative

$$d_A : \Omega^p(M, adP) \to \Omega^{p+1}(M, adP),$$

where $adP = P \times_{ad} \mathrm{Lie}\, G$ is the adjoint Lie algebra bundle. The metric g on M and the killing form K on Lie G determine an inner product $(\ ,\ )$ on $\Lambda^p T^*M \otimes adP$. This defines the L^2-inner product on $\Omega^p(M, adP)$:

$$\langle \sigma, \sigma' \rangle = \int_M (\sigma(x), \sigma'(x))\, \mathrm{d}\, vol(M, g). \tag{2.1}$$

The connection $A \in \mathcal{C}$ can also be described by its connection form

$$\omega_A : TP \to \mathrm{Lie} G.$$

The gauge group $\mathcal{G}$ acts on $\mathcal{C}$ in the obvious way. Let $\mathcal{C}/\mathcal{G}$ be the orbit space. Introducing suitable Sobolev completions one can prove that the restriction of the canonical projection $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{G}$ to the irreducible connections $\mathcal{C}^*$ is a principal $\mathcal{G}^*$-fibration, where $\mathcal{G}^* = \mathcal{G}/Z(G)$ ($Z(G)$ the centre of G) (see [13]). The subset $\mathcal{C}^*$ is open and dense in $\mathcal{C}$, hence the tangent space at $A \in \mathcal{C}^*$ is the affine space $A + \Omega^1(M, adP)$. Any element $a \in \Omega^1(M, adP)$ can be decomposed as (note that A is irreducible)

$$a = d_A(d_A^* d_A)^{-1} d_A^*(a) + (a - d_A(d_A^* d_A)^{-1} d_A^*(a)), \tag{2.2}$$

where $d_A(d_A^* d_A)^{-1} d_A^*(a)$ is tangent to the orbit of $A \in \mathcal{C}^*$, and $a - d_A(d_A^* d_A)^{-1} d_A^*(a)$ is "horizontal" with respect to $\mathcal{C}^* \to \mathcal{C}^*/\mathcal{G}^*$. In other words, (2.2) defines canonically a connection on $\mathcal{C}^* \to \mathcal{C}^*/\mathcal{G}^*$. Its connection form is denoted by

$$\Omega : T\mathcal{C}^* \to \mathrm{Lie}\, \mathcal{G}^* = \Omega^0(M, adP).$$

It is given by

$$\Omega(a) = (d_A^* d_A)^{-1} d_A^*(a). \tag{2.3}$$

For a given connection A, let $F_A \in \Omega^2(M, adP)$ be its curvature.

The Yang–Mills functional

$$L_{\mathrm{YM}} : \mathcal{C} \to \mathbb{R}$$

is defined by

$$L_{\mathrm{YM}}(A) = \frac{1}{2} \int_M (F_A, F_A)\,\mathrm{d}\,vol(M, g). \tag{2.4}$$

It is known that L_{YM} is $\mathcal{G}$-invariant, hence it can be considered as a functional on the orbit space $\mathcal{C}/\mathcal{G}$. The expectation value of an observable $\phi : \mathcal{C} \to \mathbb{R}$, i.e. of a $\mathcal{G}$-invariant function ϕ, is a quotient of path-integrals

$$\langle \phi \rangle = \frac{\int \phi(A) \mathrm{e}^{-L_{\mathrm{YM}}(A)} \mathcal{D}\mathcal{A}}{\int \mathrm{e}^{-L_{\mathrm{YM}}(A)} \mathcal{D}\mathcal{A}}, \tag{2.5}$$

where integration (in the sense of Feynman–Wiener) is over the space of paths of connections of all principal G-bundles over M. We use this quantization procedure to derive our effective Lagrangian. Let us consider the path-integral over a fixed principal G-bundle $P \to M$, i.e. over $\mathcal{C}$ (or even $\mathcal{C}^*$). Let $\mathcal{B}^* = \mathcal{C}^*/\mathcal{G}^*$. Now the splitting of

$$T_A\mathcal{C}^* = A + \Omega^1(M, ad P)$$

given by the connection Ω (see (2.3)) is L^2-orthogonal. This defines a metric on the horizontal part, i.e. on $T\mathcal{B}^*$. In analogy with the finite-dimensional case we can write

$$\int_{\mathcal{C}^*} \phi(A) \mathrm{e}^{-L_{\mathrm{YM}}(A)} \mathcal{D}\mathcal{A} = vol(\mathcal{G}) \int_{\mathcal{B}^*} \phi([A]) \mathrm{e}^{-L_{YM}([A])} (\det d_A^* d_A)^{1/2} \mathcal{D}\mathcal{B}. \tag{2.6}$$

Note that

$$-d_A : \mathrm{Lie}\,\mathcal{G} \to T_A\mathcal{C}^*$$

is the induced map of the orbit inclusion $i_A : \mathcal{G} \to \mathcal{C}$ at A. One has gained by this method that the infinite volume of $\mathcal{G}^*$ drops out in (2.5). On the other hand, the determinant of the self-adjoint operator

$$d_A^* d_A : \Omega^0(M, ad P) \to \Omega^0(M, ad P)$$

can be calculated using the zeta function

$$\sqrt{\det d_A^* d_A} = \mathrm{e}^{-(1/2)\zeta_A'(0)}. \tag{2.7}$$

Writing the total action as an exponential one is lead to make the following:

Definition 1.

$$L_{\mathrm{eff}}(A) = L_{\mathrm{YM}}(A) + \tfrac{1}{2}\zeta_A'(0)$$

is said to be the *effective Lagrange functional*.

Lemma 1. *The effective Lagrange functional*

$$L_{\mathrm{eff}} : \mathcal{C}^* \to \mathbb{R}$$

is invariant under $\mathcal{G}^*$, *i.e. induces a map*

$$L_{\mathrm{eff}} : \mathcal{C}^*/\mathcal{G}^* \to \mathbb{R}.$$

This follows immediately from Lemma 3.1(a) of [8].

With this definition one can write the path-integral in the form

$$\int_{\mathcal{B}^*} \phi([A])\mathrm{e}^{-L_{\mathrm{eff}}([A])}\mathcal{D}\mathcal{B}.$$

To proceed further, physicists apply perturbative methods. To write down the n-point Green functions, i.e. its generating functionals, the exponent $L_{\mathrm{eff}}(A)$ has to be written in the form of an integral over M (see [17,18]).

Since

$$L_{\mathrm{YM}}(A) = \frac{1}{2}\int_M (F_A, F_A)\,\mathrm{d}\,vol(M),$$

it is desirable to write $\zeta_A'(0)$ as well as an integral over M.

3. The equation of motion

We consider the effective Lagrange functional

$$L_{\mathrm{eff}} : \mathcal{C}^* \to \mathbb{R}$$

defined in Section 2. To derive its Euler–Lagrange equation one considers its differential

$$\delta L_{\mathrm{eff}} : T\mathcal{C}^* \to \mathbb{R}.$$

For a fixed $A \in \mathcal{C}^*$,

$$\delta L_{\mathrm{eff}}(A) : T_A\mathcal{C}^* \to \mathbb{R}$$

is a bounded linear map. In more detail, for any $a \in \Omega^1(M, ad P)$ we have

$$\delta L_{\mathrm{eff}}(A)(a) = \int_M (d_A^* F_A, a)\,\mathrm{d}\,vol(M) + \frac{1}{2}\delta\zeta_A'(0)(a). \tag{3.1}$$

Definition 2. Using the L^2-inner product of $\Omega^1(M, ad P)$ we define J_A by

$$\delta\zeta_A'(0)(a) = \int_M (J_A, a)\,\mathrm{d}\,vol(M).$$

Note that J_A belongs to the L^2-completion of $\Omega^1(M, ad P)$. The equation of motion can now be written in the form

$$d_A^* F_A + \tfrac{1}{2} J_A = 0. \tag{3.2}$$

The current is rather an abstract object. In order to get more information we are going to calculate $\delta\zeta_A'(0)$.

In the following we will write "Tr" for the *trace* of an operator on a space of sections and "tr" for the *trace* of an endomorphism of a single fibre.

We begin with the following formula (see [20, Section 4] or [8, Lemma A.6]):

$$\zeta_A'(0) = C\zeta_A(0) + \left[\int_0^\infty \mathrm{d}t\, t^{s-1} \mathrm{Tr}(\mathrm{e}^{-td_A^* d_A}) - \frac{\zeta_A(0)}{s} \right]_{s=0}, \tag{3.3}$$

where C is the Euler constant.

The integral

$$B_A(s) = \Gamma(s)\zeta_A(s) = \int_0^\infty \mathrm{d}t\, t^{s-1}\, \mathrm{Tr}(\mathrm{e}^{-td_A^* d_A})$$

has a simple pole at $s = 0$ with residue equal to $\zeta_A(0)$, i.e. it expands as

$$\int_0^\infty \mathrm{d}t\, t^{s-1}\, \mathrm{Tr}(\mathrm{e}^{-td_A^* d_A}) = \frac{\zeta_A(0)}{s} + b_0(A) + b_1(A)s + \cdots \tag{3.4}$$

Therefore it follows from (3.3)

$$\delta\zeta_A'(0) = C\delta\zeta_A(0) + \delta b_0(A). \tag{3.5}$$

Since $\ker(d_A^* d_A) = 0$ (because A is irreducible), we have

$$\zeta_A(0) = \frac{1}{(4\pi)^2} \int_M \mathrm{tr}\, a_2(x, x)\, \mathrm{d}\, vol(M). \tag{3.6}$$

The asymptotic expansion of the heat kernel takes the following form:

$$G(t, x, y) \sim \frac{1}{(4\pi t)^2} \mathrm{e}^{-\rho^2(x,y)/4t} \sum_{n=0}^{\infty} a_n(x, y) t^n \tag{3.7}$$

as $t \to 0$. Here $a_n(x, y)$ are sections in the bundle

$$ad P \otimes ad P^* \to M \times M$$

and

$$\rho : M \times M \to \mathbb{R}$$

is a distance function near the diagonal (we abuse notation and do not write $a_n(x, x)$ as a form or density, see [2] or [8]).

Observe that

$$ad P \otimes ad P^*|_{\Delta(M)} = \mathrm{End}(ad P),$$

where $\Delta(M) \subset M \times M$ denotes the diagonal of $M \times M$.

By Lemma 4.1(a) of [8] one has

$$\mathrm{tr}\, a_2(x, x) = 3h(x) - \tfrac{2}{3}(F_A(x), F_A(x)),$$

where $h(x)$ does not depend on A, hence it disappears in $\delta\zeta_A(0)$.

Putting this into (3.5) we obtain

$$\frac{1}{2}\delta\zeta_A'(0) = -\frac{1}{3}\frac{C}{(4\pi)^2}\,\delta\int_M (F_A, F_A)\,\mathrm{d}\,vol(M) + \frac{1}{2}\delta b_0(A). \tag{3.8}$$

If we define j_A as J_A by

$$\delta b_0(A)(a) = \int_M (j_A, a)\,\mathrm{d}\,vol(M), \tag{3.9}$$

then one obtains the following equation of motion from (3.2):

$$\left(1 - \frac{C}{24\pi^2}\right) d_A^* F_A + \frac{1}{2} j_A = 0. \tag{3.10}$$

Note that the variation of the integral in (3.8) contributes $2d_A^* F_A$. The current j_A belongs to the L^2-completion of $\Omega^1(M, ad P)$.

To calculate $\delta b_0(A)(a)$ we have to consider (by (3.4))

$$\delta B_A(s)(a) = \lim_{t \to 0} \frac{1}{t}\left(B_{A+ta}(s) - B_A(s)\right).$$

By Lemma A.5 of [8] we have

$$\delta B_A(s) = -\int_0^\infty \mathrm{d}t\, t^s \mathrm{tr}(\delta(d_A^* d_A)\mathrm{e}^{-t d_A^* d_A}) \tag{3.11}$$

with

$$\delta(d_A^* d_A)(a) = \lim_{t \to 0} \frac{1}{t}(d_{A+ta}^* d_{A+ta} - d_A^* d_A)$$

$$= d_A^* \circ [a, \cdot] - *[a, \cdot] * d_A \tag{3.12}$$

for any $a \in \Omega^1(M, ad P)$.

Here the map

$$[\cdot, \cdot] : \Omega^1(M, ad P) \times \Omega^0(M, ad P) \to \Omega^1(M, ad P)$$

is the pairing induced by the Lie brackets on the fibre. The symbol $*$ denotes the Hodge operator on forms given by the metric on M. Note that $d_A^* = - * d_A *$.

Thus we can state

Lemma 2. $-\delta b_A(0)$ *is the constant term in the* s*-expansion of*

$$\int_0^\infty \mathrm{d}t\, t^s \operatorname{Tr}(\delta(d_A^* d_A)\mathrm{e}^{-t d_A^* d_A}).$$

For $\delta(d_A^* d_A)$ we observe the following.

Lemma 3.

(1) *If* $a = d_A \eta$, *where* $\eta \in \Omega^0(M, ad P)$, *then*

$$\delta(d_A^* d_A)(a) = d_A^* d_A \circ [\eta, \cdot] - [\eta, \cdot] \circ d_A^* d_A.$$

(2) *If* $d_A^*(a) = 0$, *then*

$$\delta(d_A^* d_A)(a) = -2 * [a, \cdot] * d_A.$$

The proof follows by direct calculation. In particular, the proof of part (2) can also be found in [12] (see Lemma 2.13).

Corollary 1. *If* $a \in \operatorname{Im} d_A \subset \Omega^1(M, ad P)$, *then* $\delta b_A(0)(a) = 0$.

This follows immediately from Lemma 3(1) since $d_A^* d_A$ is a Hilbert–Schmidt operator. This corollary is of course not surprising because $\zeta_A'(0)$ is gauge invariant (see Lemma 3.1(a) of [8]).

Henceforth we have only to consider horizontal elements in $T_A \mathcal{C}^* = \Omega^1(M, ad P)$.

Let us denote

$$- * [a, \cdot] * = [a, \cdot]^*.$$

Then we have:

Corollary 2. *Let* $\mathcal{H} = \operatorname{Ker} \Omega \subset T\mathcal{C}^*$ *be the horizontal subbundle* (*see Section* 2). *Then* $-(1/2)\delta b_A(0)(a)$ *is the constant term in the* s*-expansion of*

$$\int_0^\infty \mathrm{d}t\, t^s \operatorname{Tr}([a, \cdot]^* d_A \mathrm{e}^{-t d_A^* d_A})$$

for any $a \in \mathcal{H}_A$.

Corollary 2 is related to the second fundamental form of the gauge orbit $\Theta_A = \mathcal{G}^* A \subset \mathcal{C}^*$ (see [12]).

For any $a \in T_A\mathcal{C}$ and $d_A^*(a) = 0$ (i.e. $a \in \mathcal{H}_A$) there is defined the second fundamental form

$$H_a : \Omega^0(M, ad P) \to \Omega^0(M, ad P)$$

(see 2.15 of [12]). Its regularized trace is given by

$$\operatorname{tr} H_a = \left[\int_0^\infty \mathrm{d}t\, t^s \operatorname{Tr}([a, \cdot]^* d_A \mathrm{e}^{-t d_A^* d_A}) \right]_{s=0} . \tag{3.13}$$

As explained in [12], the right-hand integral can have a pole at $s = 0$. This occurs precisely when $A \in \mathcal{C}^*$ is not a Yang–Mills connection.

So we can reformulate Corollary 2 as follows.

Corollary 3. *The value $\delta b_A(0)(a)$ for any $A \in \mathcal{C}^*$ and $a \in \mathcal{H}_A$ is equal to the regular part of the trace of the second fundamental form H_a of the gauge orbit $\Theta_A \subset \mathcal{C}^*$. Moreover, minimal gauge orbits $\Theta_A \subset \mathcal{C}^*$, where A is a Yang–Mills connection, are solutions of the equation of motion* (3.2).

We will now consider another aspect of $\delta b_A(0)$ and do a partial integration on (3.11). Putting $u = t^s$ and

$$v' = \operatorname{Tr}(\delta(d_A^* d_A) \mathrm{e}^{-t d_A^* d_A}),$$

one gets

$$\begin{aligned} -\delta B_A(s) = {} & [t^s \operatorname{Tr}(\delta(d_A^* d_A)(d_A^* d_A)^{-1} \mathrm{e}^{-t d_A^* d_A})]_0^\infty \\ & - s \int_0^\infty \mathrm{d}t\, t^{s-1} \operatorname{Tr}(\delta(d_A^* d_A)(d_A^* d_A)^{-1} \mathrm{e}^{-t d_A^* d_A}). \end{aligned} \tag{3.14}$$

This can be done for $\operatorname{Re}(s)$ sufficiently large. The first summand vanishes, so we have the following result.

Lemma 4. *$\delta b_A(0)$ is the residue at $s = 0$ of the meromorphic extension of the integral*

$$\int_0^\infty \mathrm{d}t\, t^{s-1} \operatorname{Tr}(\delta(d_A^* d_A)(d_A^* d_A)^{-1} \mathrm{e}^{-t d_A^* d_A}).$$

For the meromorphic extension it suffices to consider the integral in the equation of Lemma 4 between 0 and 1 (a similar proof as for Lemma A.1 of [8]). If we now expand the integrand under $\int_0^1$ in a series in t, then we obtain:

Lemma 5. *$\delta b_A(0)$ is the constant term in the t-expansion of*

$$\operatorname{Tr}(\delta(d_A^* d_A)(d_A^* d_A)^{-1} \mathrm{e}^{-t d_A^* d_A}).$$

This is the starting point for further calculations. We will explain it in more detail in the next section.

4. On the polarization current

We continue to study the current $j_A(x)$ defined by (3.9).

From Lemmas 3 and 5 we obtain that

$$\int_M (j_A, a)\,\mathrm{d}\,vol(M) = \delta b_A(0)(a)$$

$$= -2\,\text{constant term in the } t\text{-expansion of } \operatorname{Tr}(*[a,\cdot] * d_A(d_A^* d_A)^{-1} \mathrm{e}^{-t d_A^* d_A}). \qquad (4.1)$$

Now observe that

$$(d_A^* d_A)^{-1} \mathrm{e}^{-t d_A^* d_A} = \mathrm{e}^{-t d_A^* d_A} (d_A^* d_A)^{-1}.$$

We make use of this commutation and note that the heat kernel $G_A(x, y, t)$ of $d_A^* d_A$ is smooth. Therefore $d_A^x G_A(x, y, t)$ is the kernel function of $d_A \mathrm{e}^{-t d_A^* d_A}$. The upper index x denotes the *partial covariant derivative* with respect to the x-variable.

Let $G_A(x, y)$ be the Green function of $d_A^* d_A$.

Then we get

$$\operatorname{Tr}(*[a,\cdot] * d_A(d_A^* d_A)^{-1} \mathrm{e}^{-t d_A^* d_A})$$

$$= \int_M \left(\int_M \operatorname{tr}(*[a(x),\cdot] * d_A^x G_A(x, y, t) G_A(y, x))\,\mathrm{d}y \right) \mathrm{d}x. \qquad (4.2)$$

Here $\mathrm{d}x$ (resp. $\mathrm{d}y$) is $\mathrm{d}\,vol(M)$, i.e. the infinitesimal volume element of M.

Now recall that the scalar product of Lie$G$ is the negative of the killing form.

Then (4.1) and (4.2) imply

$$[j_A(x),\cdot] = 2\ c.t. \int_M d_A^x G_A(x, y, t) G_A(y, x)\,\mathrm{d}y, \qquad (4.3)$$

where $c.t.$ stands for "constant term" in the t-expansion.

Remark 1. The value of (4.2) is the difference in traces of two trace class operators. To see this, let us denote $A' = A + a$.

Then we have

$$*[a,\cdot] * d_A = d_{A'}^* d_A - d_A^* d_A.$$

Let $P_t = \mathrm{e}^{-t d_A^* d_A}$ and $Q_s = (d_A^* d_A)^{-s}$. Then P_t is a Hilbert–Schmidt operator (in fact it is of trace class), hence $P_t Q_s$ is Hilbert–Schmidt.

Moreover, we have

$$P_t Q_s = Q_s P_t,$$

hence

$$P_t Q_s = (P_{t/2} Q_{s/2})(P_{t/2} Q_{s/2}),$$

i.e. $P_t Q_s$ is of trace class too. The operator $d^*_{A'} d_A$ has a smooth kernel, so it is also Hilbert–Schmidt. Therefore, the composition $d^*_{A'} d_A P_t Q_s$ is of trace class too.

Its kernel function is

$$\mathcal{U}_a(x, w, t, s) = \iint\limits_M K_a(x, y) G_A(y, z, t) L(z, w, s)\, \mathrm{d}y\, \mathrm{d}z,$$

where K_a and $L(\cdot, \cdot, s)$ are the kernels of $d^*_{A'} d_A$ and Q_s, respectively.

Therefore, we have

$$\mathrm{Tr}(d^*_{A'} d_A P_t Q_s) = \int\limits_M \mathrm{tr}\, \mathcal{U}_a(x, x, t, s)\, \mathrm{d}x$$

and hence (4.2) becomes

$$\int\limits_M \mathrm{tr}\, \mathcal{U}_a(x, x, t, 1)\, \mathrm{d}x - \int\limits_M \mathrm{tr}\, \mathcal{U}_0(x, x, t, 1)\, \mathrm{d}x.$$

Observe that $\mathcal{U}_0(x, y, t, 1) = G_A(x, y, t)$.

Now we will calculate $[j_A(x), \cdot]$. This requires some preliminary calculations.

Let us begin by calculating $d^x_A G_A(x, y, t)$.

From (3.7) we get

$$\begin{aligned} & d^x_A \left(\frac{1}{(4\pi t)^2} \mathrm{e}^{-\rho^2(x,y)/4t} \sum_{n=0}^{\infty} a_n(x, y) t^n \right) \\ & \quad = \partial^x_\mu \left(\frac{1}{(4\pi t)^2} \mathrm{e}^{-\rho^2(x,y)/4t} \right) \mathrm{d}x^\mu \sum_{n=0}^{\infty} a_n(x, y) t^n \\ & \qquad + \frac{1}{(4\pi t)^2} \mathrm{e}^{-\rho^2(x,y)/4t} \sum_{n=0}^{\infty} d^x_A a_n(x, y) t^n, \end{aligned} \tag{4.4}$$

where

$$\partial^x_\mu \left(\frac{1}{(4\pi t)^2} \mathrm{e}^{-\rho^2(x,y)/4t} \right) = -\frac{1}{4t} \frac{1}{(4\pi t)^2} \partial^x_\mu \rho^2(x, y) \mathrm{e}^{-\rho^2(x,y)/4t}.$$

Here we have used the notation

$$d_A = (\partial_\mu + A_\mu)\, \mathrm{d}x^\mu.$$

Now recall that

$$a_0(x, y) = \tau(x, y) : (adP)_y \to (adP)_x$$

is the *parallel transport* of A.

Using the Mellin transformation

$$L_s(x, y) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} G_A(x, y, t),$$

one can calculate $G_A(x, y) = L_1(x, y)$.

Lemma 6. *For $\rho(x, y) < r$, where r is the injectivity radius, one has*

$$G_A(x, y) = \frac{1}{(4\pi)^2} \left(\frac{4a_0(x, y)}{\rho^2(x, y)} - a_1(x, y) \log \rho^2(x, y) \right) + R_A(x, y),$$

where $R_A(x, y)$ is a smooth function.

For a proof we refer to [14, Theorem 2.2].

We have to calculate the constant terms in the asymptotic t-expansions of all possible products of (4.4) with $G_A(x, y)$.

For this we need formulae for asymptotic expansions of type

$$\frac{1}{(4\pi t)^2} \int_{\mathbb{R}^4} \varphi(y) \mathrm{e}^{-\rho^2(x,y)/4t}\, \mathrm{d}y$$

and

$$\frac{1}{(4\pi t)^2} \int_{\mathbb{R}^4} \varphi(y) \log \rho(x, y) \mathrm{e}^{-\rho^2(x,y)/4t}\, \mathrm{d}y.$$

Here $\varphi : \mathbb{R}^4 \to V$ is a smooth map into a vector space V. In a normal coordinate system the metric g is standard modulo $\|x - y\|^2$. It will be sufficient to consider the case $\rho(x, y) = \|x - y\|$. Applying the cut-off procedure our applications regard maps φ which are zero outside a small ball around a point x. By taking components of φ we can assume $V = \mathbb{R}$.

Lemma 7. *Let $\varphi : \mathbb{R}^4 \to V = \mathbb{R}$ be a smooth map such that $\varphi(x) = 0$. Then we have*

$$\frac{1}{(4\pi t)^2} \int_{\mathbb{R}^4} \varphi(y) \log \|x - y\|\, \mathrm{e}^{-\|x-y\|^2/4t}\, \mathrm{d}y$$

$$\sim \sum_{|\alpha|=1}^{\infty} b_\alpha \varphi^{(\alpha)}(x) \frac{t^{|\alpha|/2}}{\alpha!} + \log \sqrt{t} \sum_{|\alpha|=1}^{\infty} c_\alpha \varphi^{(\alpha)}(x) \frac{t^{|\alpha|/2}}{\alpha!}.$$

Here $\alpha = (\alpha_1, \alpha_2, \ldots)$, $|\alpha|$, $\varphi^{(\alpha)}$, and $\alpha!$ represent the usual multi-index notation.

The constants b_α and c_α can be calculated:

$$b_\alpha = \frac{1}{(4\pi)^2} \int_{\mathbb{R}^4} e^{-\|v\|^2/4}\, v^\alpha \log \|v\| \; dv$$

$$c_\alpha = \frac{1}{(4\pi)^2} \int_{\mathbb{R}^4} e^{-\|v\|^2/4}\, v^\alpha \, dv.$$

Proof. Making the standard substitution $y - x = \sqrt{t}\, v$, one obtains

$$\begin{aligned} &\frac{1}{(4\pi t)^2} \int_{\mathbb{R}^4} \varphi(y) \log \|x - y\|\, e^{-\|x-y\|^2/4t}\, dy \\ &\quad = \frac{1}{(4\pi)^2} \int_{\mathbb{R}^4} \varphi(x + \sqrt{t}\, v) \log \sqrt{t} e^{-\|v\|^2/4}\, dv \\ &\qquad + \frac{1}{(4\pi)^2} \int_{\mathbb{R}^4} \varphi(x + \sqrt{t}\, v) \log \|v\|\, e^{-\|v\|^2/4}\, dv. \end{aligned}$$

Now we have

$$\varphi(x + \sqrt{t}\, v) \sim \sum_{|\alpha|=1}^{\infty} v^\alpha \varphi^{(\alpha)}(x) \frac{t^{|\alpha|/2}}{\alpha!}.$$

The result then follows. To avoid the term

$$b_0 = \frac{1}{(4\pi)^2} \int_{\mathbb{R}^4} \varphi(v) \log \|v\|\, e^{-\|v\|^2/4},$$

which might not exist, we have used the assumption $\varphi(x) = 0$. □

Similarly one proves:

Lemma 8. *Let $\varphi : \mathbb{R}^4 \to V = \mathbb{R}$ be smooth. Then we have*

$$\frac{1}{(4\pi t)^2} \int_{\mathbb{R}^4} \varphi(y) e^{-\|x-y\|^2/4t}\, dy \sim \sum_{|\alpha|=0}^{\infty} \varphi^{(\alpha)}(x) c_\alpha \frac{t^{|\alpha|/2}}{\alpha!}$$

with c_α as above and $c_0 = 1$.

The following lemma is well-known.

Lemma 9.

(1) $$\frac{1}{(4\pi t)^2} \int_{\mathbb{R}^4} \frac{v^\alpha}{\|v\|^{2L}} e^{-\|v\|^2/4t}\, dv$$

is finite of order $t^{|\alpha|/2-L}$ if $|\alpha|/2 - L > -2$. It is zero if $\alpha \neq 2\beta$.

(2) $$\frac{1}{(4\pi t)^2}\int_{\mathbb{R}^4}\frac{v^\mu v^\nu}{\|v\|^2}\mathrm{e}^{-\|v\|^2/4t}\,\mathrm{d}v = \delta_{\mu\nu}.$$

We must also calculate $d_A^x\tau(x, y)$. The proof is lengthy so it will be given in Appendix B.

Proposition 1. *Let*

$$d_A\tau(x, y) = \partial_\lambda^A\tau(x, y)\,\mathrm{d}x^\lambda.$$

Then, in the normal coordinate system around $x \equiv (x^i)$, the following expansion holds:

$$\begin{aligned}\partial_\lambda^A\tau(x, y) = &-\tfrac{1}{2}[F_{\lambda s}(x)(y^s - x^s), \cdot]\\ &- \tfrac{1}{6}[\partial_t^A F_{\lambda s}(x)(y^s - x^s)(y^t - x^t), \cdot] + \mathrm{o}(\| x - y \|^3).\end{aligned}$$

As a consequence of Lemma 7 we obtain the following.

Corollary 4. *Suppose that $\varphi : \mathbb{R}^4 \to V = \mathbb{R}$ is smooth. The following holds:*
(a) *If $\varphi(x) = 0$, then the constant term in the t-expansion of*

$$\frac{1}{(4\pi t)^2}\int_{\mathbb{R}^4}\varphi(y)\log\rho(x, y)\mathrm{e}^{-\rho^2(x,y)/4t}\,\mathrm{d}y$$

is zero;
(b) *If $\varphi(x) = 0$, then the t-linear term in the asymptotic t-expansion of*

$$\frac{1}{(4\pi t)^2}\int_{\mathbb{R}^4}\varphi(y)\log\rho(x, y)\mathrm{e}^{-\rho^2(x,y)/4t}\,\mathrm{d}y$$

is equal to

$$\frac{1}{2}\sum b_{(\mu,\nu)}\frac{\partial^2\varphi}{\partial x^\mu\partial x^\nu}(x) = \frac{1}{2}b_{(1,1)}\sum\frac{\partial^2\varphi}{\partial^2 x^\mu}(x).$$

Proof. (a) Follows immediately. For (b) we have to calculate the constant term of

$$\frac{1}{t}\left(\log\sqrt{t}\sum_{|\alpha|\geq 1}c_\alpha\varphi^{(\alpha)}(x)\frac{t^{|\alpha|/2}}{\alpha!} + \sum_{|\alpha|\geq 1}b_\alpha\varphi^{(\alpha)}(x)\frac{t^{|\alpha|/2}}{\alpha!}\right).$$

Since $t^{|\alpha|/2}\log\sqrt{t} \to 0$ as $t \to 0$ whenever $|\alpha| > 0$, there remains only

$$\sum_{|\alpha|=2}b_\alpha\varphi^{(\alpha)}(x)\frac{1}{\alpha!}.$$

But by symmetry reasons we have $b_{(\mu,\nu)} = 0$ for $\mu \neq \nu$, and

$$b_{(\mu,\mu)} = b_{(1,1)} = b.$$

Observe that for the proof of (b) the assumption $\varphi(x) = 0$ is essential. The coefficient $b = b_{(1,1)}$ will be calculated in Appendix A.

Finally we recall that (see (A.12) of [12])

$$d_A^x \tau(x,y)|_{y=x} = 0. \tag{4.5}$$

We have now all ingredients for our calculation of $[j_A(x), \cdot]$.

We begin with the contribution of $R_A(x,y)$, i.e.

$$2\ c.t. \int_M d_A G_A(x,y,t) R_A(y,x)\,\mathrm{d}y.$$

Since we are interested in small t-values, we can apply the usual cut-off procedure and assume $G_A(x,y,t) = 0$ outside the normal neighbourhood $B(x,r)$. So the integral can be transformed into the disc of radius r in $\mathbb{R}^4$. The transformation $y - x = \sqrt{t}\,v$ changes the integral into an integral over a disc of radius $\sqrt{r}/t$ which becomes an integral over $\mathbb{R}^4$ as $t \to 0$. Hence we can apply the above asymptotic expansions.

Using (4.4) we have to consider

$$2\frac{t^n}{(4\pi t)^2} \int_{\mathbb{R}^4} d_A^x a_n(x,y) R_A(y,x) \mathrm{e}^{-\|x-y\|^2/4t}\,\mathrm{d}y$$

as $t \to 0$. This gives zero for $n > 0$ by Lemma 8.

For $n = 0$ we obtain by (4.5)

$$2\frac{1}{(4\pi t)^2} \int_{\mathbb{R}^4} d_A^x a_0(x,y) R_A(y,x) \mathrm{e}^{-\|x-y\|^2/4t}\,\mathrm{d}y = 2 d_A^x a_0(x,y) R_A(y,x)|_{y=x} = 0.$$

Now consider the other part of $d_A G_A(x,y,t)$:

$$-2\frac{1}{4t}\frac{t^n}{(4\pi t)^2} \int_{\mathbb{R}^4} a_n(x,y) R_A(y,x) \partial_\mu^x \rho^2(x,y) \mathrm{e}^{-\rho^2(x,y)/4t}\,\mathrm{d}y$$

as $t \to 0$.

Let

$$\varphi(y) = a_n(x,y) R_A(y,x) \partial_\mu^x \rho^2(x,y).$$

Since $\varphi(x) = 0$, the only relevant term comes from $n = 0$.

By Lemma 8 it is equal to

$$-2\frac{1}{4} \sum_{|\alpha|=2} \frac{\varphi^{(\alpha)}(x)}{\alpha!},$$

where

$$\varphi(y) = a_0(x,y) R_A(y,x) \partial_\mu^x \rho^2(x,y).$$

Since $d_A \tau(x,y)|_{y=x} = 0$, one has

$$a_0(x,y) = \mathrm{Id} + \mathrm{o}(\|x-y\|^2),$$

hence it follows that

$$
\begin{aligned}
&2\sum_{\mu}\left(-\frac{1}{4}\sum_{|\alpha|=2}\frac{\varphi^{(\alpha)}(x)}{\alpha!}\right)\mathrm{d}x^{\mu} \\
&\quad=-\sum_{\alpha=(\nu,\mu)}\frac{c_{\alpha}}{\alpha!}\partial_{\nu}R_{A}(x,x)\,\mathrm{d}x^{\mu} \\
&\quad=-\frac{1}{2}c_{(1,1)}\sum_{\mu}\partial_{\mu}R_{A}(x,x)\,\mathrm{d}x^{\mu}=-\frac{1}{2}c_{(1,1)}\,\mathrm{d}R_{A}(x,x).
\end{aligned}
$$

Since

$$
c_{(1,1)}=\frac{1}{(4\pi)^{2}}\int_{\mathbb{R}^{4}}v_{1}^{2}\mathrm{e}^{-\|v\|^{2}/4}\,\mathrm{d}v=2,
$$

one obtains

$$
2\ c.t.\int_{M}d_{A}G_{A}(x,y,t)R_{A}(y,x)\,\mathrm{d}y=-\mathrm{d}R_{A}(x,x). \tag{4.6}
$$

Next we calculate the contribution of

$$
-\frac{1}{(4\pi)^{2}}a_{1}(x,y)\log\rho^{2}(x,y),
$$

i.e.

$$
2\ c.t.\left(-\frac{2}{(4\pi)^{2}}\int_{M}d_{A}G_{A}(x,y,t)a_{1}(y,x)\log\rho(x,y)\,\mathrm{d}y\right).
$$

Using (4.4) we have to consider

$$
-\frac{4t^{n}}{(4\pi)^{2}}\int_{\mathbb{R}^{4}}d_{A}^{x}a_{n}(x,y)a_{1}(y,x)\log\rho(x,y)\mathrm{e}^{-\rho^{2}(x,y)/4t}\,\mathrm{d}y
$$

as $t\to 0$.

We set

$$
\varphi(y)=d_{A}^{x}a_{n}(x,y)a_{1}(y,x).
$$

By hypothesis that the scalar curvature vanishes, we have $a_{1}(x,x)=0$, i.e. $\varphi(x)=0$. We have no contribution for $n=0$ by Corollary 4(a).

The other part is

$$
\frac{4}{(4\pi)^{2}}\frac{1}{4t}\frac{t^{n}}{(4\pi t)^{2}}\int_{\mathbb{R}^{4}}a_{n}(x,y)a_{1}(y,x)\partial_{\mu}^{x}\rho^{2}(x,y)\log\rho(x,y)\mathrm{e}^{-\rho^{2}(x,y)/4t}\,\mathrm{d}y.
$$

Putting

$$
\varphi(y)=a_{n}(x,y)a_{1}(y,x)\partial_{\mu}^{x}\rho^{2}(x,y),
$$

we have obviously that $\varphi(x) = 0$. There are only two relevant values for n which contribute, namely $n = 0$ and $n = 1$.

For $n = 1$ we can apply Corollary 4(a) to get zero.

For $n = 0$ we get from Corollary 4(b):

$$
\begin{aligned}
&2\, c.t. \frac{-2}{(4\pi)^2} \int_M d_A G_A(x, y, t) a_1(y, x) \log \rho(x, y)\, \mathrm{d}y \\
&= \frac{4}{(4\pi)^2} \frac{1}{4} \frac{1}{2} b_{(1,1)} \sum_{\mu=1}^{4} \frac{\partial^2 \varphi}{\partial^2 x^\mu}(x)\, \mathrm{d}x^\mu \\
&= \frac{b_{(1,1)}}{2} \frac{1}{(4\pi)^2} 2\, \mathrm{d}a_1(x, x) = \frac{b_{(1,1)}}{(4\pi)^2}\, \mathrm{d}a_1(x, x).
\end{aligned}
\tag{4.7}
$$

Note that the factor 2 in front of $\mathrm{d}a_1(x, x)$ comes from $\partial_\mu^x \rho^2(x, y)$. We will calculate $b = b_{(1,1)}$ in Appendix A.

It remains to calculate

$$
2\, c.t. \frac{4}{(4\pi)^2} \int_M d_A^x G_A(x, y, t) a_0(y, x) \rho^{-2}(x, y)\, \mathrm{d}y.
$$

According to (4.4) we have to consider

$$
2 \left(-\frac{4}{(4\pi)^2} \frac{1}{4t} \frac{t^n}{(4\pi t)^2} \int_{\mathbb{R}^4} a_n(x, y) a_0(y, x) \frac{\partial_\mu^x \rho^2(x, y)}{\rho^2(x, y)} \mathrm{e}^{-\rho^2(x,y)/4t}\, \mathrm{d}y \right).
$$

If we look at the Taylor expansion of

$$
\varphi(y) = a_n(x, y) a_0(y, x) \partial_\mu^x \rho^2(x, y),
$$

it will be of the type

$$
\varphi(y) = 2(y^\mu - x^\mu)(a_n(x, x) a_0(x, x) + \cdots),
$$

where in the larger parenthesis is indicated the Taylor expansion of $a_n(x, y) a_0(y, x)$. Note that for $n = 0$ we have $a_0(x, y) a_0(y, x) = \mathrm{Id}$, hence the integral vanishes by Lemma 9(1). Again, it follows from Lemma 9(1) that non-zero contributions can only occur for $n = 1$. In this case, since $a_0(x, y) = \mathrm{Id} + \mathrm{o}(\|x - y\|^2)$ and $a_1(x, x) = 0$, the Taylor expansion is of the type

$$
\varphi(y) = 2(y^\mu - x^\mu) \left(\sum_\nu a_1^{(\nu)}(x, x)(y^\nu - x^\nu) + \cdots \right).
$$

The higher terms do not contribute by Lemma 9.

Then we obtain by Lemma 9(2)

$$
-2 \frac{4}{(4\pi)^2} \frac{1}{2} \partial_\mu^x a_1(x, x)\, \mathrm{d}x^\mu = -\frac{4}{(4\pi)^2} d a_1(x, x).
\tag{4.8}
$$

It remains to consider

$$2\frac{4}{(4\pi)^2}\frac{t^n}{(4\pi t)^2}\int\limits_{\mathbb{R}^4} d_A^x a_n(x,y)a_0(y,x)\rho^{-2}(x,y)\mathrm{e}^{-\rho^2(x,y)/4t}\,\mathrm{d}y.$$

Again, it can be seen that only $n = 0$ gives a contribution.

We apply Proposition 1 to

$$d_A^x a_0(x,y) = \partial_\lambda^A a_0(x,y)\,\mathrm{d}x^\lambda.$$

Since $a_0(x,y) = \mathrm{Id} + \mathrm{o}(\|x-y\|^2)$, only the quadratic term in the expansion of $d_A a_0$ contributes by Lemma 9.

Thus one gets

$$2\frac{4}{(4\pi)^2}\left(-\frac{1}{6}\right)\sum_{t,\lambda}\partial_t^A F_{\lambda t}(x)\,\mathrm{d}x^\lambda = -\frac{1}{12\pi^2}d_A^* F_A(x). \tag{4.9}$$

Remark 2. In Lemmas 7 and 8 the functions $\varphi : \mathbb{R}^4 \to V$ are local representations of globally defined sections. For example,

$$\varphi(y) = a_0(x,y)a_1(y,x) : U_x \to (adP)_x \otimes (adP)_x^*,$$

where U_x is a neighbourhood of x in M. However, to trivialize $adP|_{U_x}$ we use the parallel transport $\tau(x,y) : (adP)_y \to (adP)_x$. Hence to localize $a_j(x,y)$, one has to form the composition with $\tau(x,y)$. Moreover, in U_x we take the normal coordinates $\exp_x y) = y$. As a consequence, the partial derivative $\partial_\mu a_1$, for example, is the partial derivative of $\partial_\mu(\tau \circ a_1)$ which equals $\partial_\mu^A a_1$ by Lemma (A.9) of [12], where

$$\partial_\mu^A = \partial_\mu + [A_\mu, \cdot].$$

This means that da_1 becomes $d_A a_1$ in a coordinate free description. The same holds for R_A.

Putting together (4.6)–(4.9) we obtain the following result.

Theorem 2. *If M is a closed connected orientable Riemannian 4-manifold of scalar curvature zero and $P \to M$ is a principal $SU(2)$-bundle, then the current j_A, defined by (3.9), satisfies the formula:*

$$[j_A(x),\cdot] = -\frac{1}{12\pi^2}[d_A^* F_A(x),\cdot] + \frac{b-4}{(4\pi)^2}d_A a_1(x,x) - d_A R_A(x,x),$$

where

$$b = b_{(1,1)} = \frac{1}{(4\pi)^2}\int\limits_{\mathbb{R}^4} v_1^2 \mathrm{e}^{-\|v\|^2/4}\log\|v\|\,\mathrm{d}v = \frac{3}{2} - C + \log 4.$$

Together with formula (3.8) we have

$$
\begin{aligned}
J_A &= -\frac{4}{3}\frac{C}{(4\pi)^2} d_A^* F_A + j_A \\
&= -\frac{1+C}{12\pi^2} d_A^* F_A + \frac{b-4}{(4\pi)^2} d_A a_1 - d_A R_A,
\end{aligned} \tag{4.10}
$$

proving our main result.

The equation of motion becomes

$$
\left(1 - \frac{1+C}{24\pi^2}\right) d_A^* F_A + \frac{b-4}{32\pi^2} d_A a_1 - \frac{1}{2} d_A R_A = 0. \tag{4.11}
$$

Remark 3. For irreducible connections A we have calculated the current J_A under the hypothesis that the scalar curvature of the underlying Riemannian manifold is zero. The current J_A belongs to the L^2-completion of $\Omega^1(M, ad P)$. However, the calculation is done in $\Omega^1(M, \mathrm{End}(ad P))$, and therefore $d_A^* F_A$ means $[d_A^* F_A, \cdot\,]$.

Appendix A

We are going to calculate

$$
b = b_{(1,1)} = \frac{1}{(4\pi)^2} \int_{\mathbb{R}^4} v_1^2 \mathrm{e}^{-\|v\|^2/4} \log \|v\| \ \mathrm{d}v.
$$

By symmetry, we have

$$
\begin{aligned}
b_{(1,1)} &= \frac{1}{4}\frac{1}{(4\pi)^2} \int_{\mathbb{R}^4} \|v\|^2 \mathrm{e}^{-\|v\|^2/4} \log \|v\| \ \mathrm{d}v \\
&= \frac{1}{4}\frac{1}{(4\pi)^2} \int_0^\infty r^2 \mathrm{e}^{-r^2/4} S_r \log r \ \mathrm{d}r,
\end{aligned}
$$

where

$$
S_r = \text{volume}\,(\mathbb{S}_r^3 = \{u \in \mathbb{R}^4 : \|u\|^2 = r^2\}) = 2\pi^2 r^3.
$$

Therefore, it follows

$$
\begin{aligned}
b_{(1,1)} &= \frac{1}{4}\frac{2\pi^2}{(4\pi)^2} \int_0^\infty r^5 \mathrm{e}^{-r^2/4} \log r \, \mathrm{d}r \\
&= \frac{2}{4^3} \int_0^\infty r^5 \mathrm{e}^{-r^2/4} \log r \ \mathrm{d}r.
\end{aligned}
$$

Using a formula on p. 527 of [15], one gets

$$
\begin{aligned}
b_{(1,1)} &= \frac{2}{4^3}\frac{1}{4(1/4)^3}\Gamma(3)\,(\psi(3) - \log 1/4)\\
&= -C + 1 + 1/2 + \log 4 = 3/2 - C + \log 4.
\end{aligned}
$$

Appendix B

In this section we are going to prove Proposition 1. In the appendix of [12] the Taylor expansion of the parallel transport was explicitly calculated:

$$\tau(x, y) : (adP)_y \to (adP)_x.$$

It can be written in the form

$$
\begin{aligned}
\tau(x, y) &= \mathrm{Id} + A_\mu(y)(x^\mu - y^\mu) + \frac{1}{2}\partial_\nu A_\mu(y)(x^\mu - y^\mu)(x^\nu - y^\nu)\\
&+ \frac{1}{3!}\partial_\gamma\partial_\nu A_\mu(y)(x^\mu - y^\mu)(x^\nu - y^\nu)(x^\gamma - y^\gamma) + \text{higher orders.}
\end{aligned}
\tag{B.1}
$$

Here it is understood that (x^μ) and (y^μ) are the normal coordinates in a normal coordinate system centred at a fixed point z. Moreover, the right-hand side of the formula acts on $(adP)_y$ via the adjoint representation, i.e. via Lie brackets. This can be seen by inserting the detailed calculations made in [12] in formula (A.7).

On the other hand, from Proposition 1.18 of [2] we obtain

$$A_\mu(y) = -\tfrac{1}{2}F_{\mu k}(x)(y^k - x^k) - \tfrac{1}{3}\partial_l F_{\mu k}(x)(y^k - x^k)(y^l - x^l) + \text{ higher orders.} \tag{B.2}$$

This formula holds in a normal coordinate system centred at the point x.

To obtain our result we proceed as follows. We calculate $\partial_\lambda^A \tau(x, y)$ with respect to the x-variable. Then we put $z = x$, and use formula (B.2). Since $A_\mu(x) = 0$, it will be enough to calculate $\partial_\lambda \tau(x, y)$. It can also be seen that the 3rd-order term of (B.1) is not relevant to us.

Thus we have to consider

$$
\begin{aligned}
&\partial_\lambda(A_\mu(y)(x^\mu - y^\mu)) = A_\lambda(y),\\
&\partial_\lambda(\tfrac{1}{2}\partial_\nu A_\mu(y)(x^\mu - y^\mu)(x^\nu - y^\nu))\\
&\quad = \tfrac{1}{2}\partial_\nu A_\lambda(y)(x^\nu - y^\nu) + \tfrac{1}{2}\partial_\lambda A_\mu(y)(x^\mu - y^\mu).
\end{aligned}
\tag{B.3}
$$

Now we put $z = x$, and use (B.2) to calculate

$$
\begin{aligned}
\partial_\nu A_\lambda(y) &= -\tfrac{1}{2}F_{\lambda\nu}(x) - \tfrac{1}{3}(\partial_l F_{\lambda\nu}(x)(y^l - x^l) + \partial_\nu F_{\lambda k}(x)(y^k - x^k)),\\
\partial_\lambda A_\mu(y) &= -\tfrac{1}{2}F_{\mu\lambda}(x) - \tfrac{1}{3}(\partial_l F_{\mu\lambda}(x)(y^l - x^l) + \partial_\lambda F_{\mu k}(x)(y^k - x^k)).
\end{aligned}
\tag{B.4}
$$

Inserting (B.4) into (B.3) we obtain

$$\partial_\lambda(\tfrac{1}{2}\partial_\nu A_\mu(y)(x^\mu - y^\mu)(x^\nu - y^\nu))$$

$$
\begin{aligned}
&= -\tfrac{1}{2}[\tfrac{1}{2}F_{\lambda\nu}(x)(x^{\nu}-y^{\nu}) + \tfrac{1}{3}\partial_l F_{\lambda\nu}(x)(y^l - x^l)(x^{\nu}-y^{\nu}) \\
&\quad + \tfrac{1}{3}\partial_{\nu} F_{\lambda k}(x)(y^k - x^k)(x^{\nu}-y^{\nu}) + \tfrac{1}{2}F_{\mu\lambda}(x)(x^{\mu}-y^{\mu}) \\
&\quad + \tfrac{1}{3}\partial_l F_{\mu\lambda}(x)(y^l - x^l)(x^{\mu}-y^{\mu}) + \tfrac{1}{3}\partial_{\lambda} F_{\mu k}(x)(y^k - x^k)(x^{\mu}-y^{\mu})] \\
&= -\tfrac{1}{6}\partial_{\nu} F_{\lambda k}(x)(y^k - x^k)(x^{\nu}-y^{\nu}) \\
&= \ \tfrac{1}{6}\partial_{\nu} F_{\lambda k}(x)(y^k - x^k)(y^{\nu}-x^{\nu}).
\end{aligned}
$$

This can be easily seen observing that $F_{rs}(x) = -F_{sr}(x)$. Therefore the last term vanishes while the first and the fourth term cancel out. The remaining three terms are equal up to sign, so two of them cancel out.

Using (B.2) again, we obtain

$$
\begin{aligned}
\partial_{\lambda}^{A}\tau(x,y) &= A_{\lambda}(y) + \tfrac{1}{6}F_{\lambda k}(x)(y^k - x^k)(y^{\nu}-x^{\nu}) + \text{higher orders} \\
&= -\tfrac{1}{2}F_{\lambda k}(x)(y^k - x^k) - \tfrac{1}{6}\partial_{\nu}F_{\lambda k}(x)(y^k - x^k)(y^{\nu}-x^{\nu}) + \text{higher orders}.
\end{aligned}
$$

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